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General Correlation Functions in the Schwinger Model at Zero and Finite Temperature

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Abstract

The general correlations between massless fermions are calculated in the Schwinger model at arbitrary temperature. The zero temperature calculations on the plane are reviewed and clarified. Then the finite temperature fermionic Green's function is computed and the results on the torus are compared to those on the plane. It is concluded that a simpler way to calculate the finite temperature results is to associate certain terms in the zero temperature structure with their finite temperature counterparts.

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1. Motivation

A testimony to the complexity of QCD is in the many outstanding problems of the theory that are still not solved. For example, it is not fundamentally understood how quarks are confined or chiral symmetry is broken, though nature seems to adhere to these properties. Many have studied the Schwinger model (QED in two Euclidean dimensions with a massless fermion) [2, 3, 4] in the hopes of gaining intuition for tackling various such problems.

Schwinger [1] proposed this model to show that gauge invariance of a vector field need not imply the existence of a massless particle. The gauge field phase shifts the right and left handed massless fermions with respect to one another causing a breakdown of chiral symmetry. This leads to an anomaly which takes form as the generation of a photon mass ($m = g/\sqrt{\pi}$). The vacuum has a non-trivial topology analogous to the case in QCD. This theory also has charge confinement because of Gauss' law in one space dimension.

Since the Schwinger model is exactly solvable, having a form for the general n-point correlation function would be useful in many calculations. The result has been approached on the plane [5], on the sphere [6], and on the torus [7]. Compact surfaces are preferred in calculations because the infinities are kept under control. Fermions on the sphere are difficult to define and for finite temperature considerations the boundary conditions of the torus are more natural. Therefore, work on the torus is desirable.

Interest in finite temperature calculations in field theories have led to the evaluation of two- and four-point correlation functions [9]. In this paper, we will provide explicit calculations for a six-point correlation function at finite temperature. When combined with the previous results, the present result yields the generic form for the n-point correlation function for the Schwinger model at finite temperature. Exact finite temperature calculations of correlation function in field theory are of paramount importance for the understanding of finite temperature spectral functions, reflecting on the effects of matter on the fundamental degrees of freedom. Real- and Euclidean-time correlators offer important information on fundamental issues related to symmetry restoration.

Our paper is organized as follows: in section 2, we give a brief analysis of the two-dimensional Euclidean Dirac operator in the plane. In section 3, we consider n-point scalar correlation functions in the two-dimensional Euclidean plane. In section 4, the same analysis is extended to the Euclidean two-dimensional torus, and the three-point scalar correlator is explicitly worked out. The result is extended to the n-point scalar correlator at finite temperature. A brief discussion is given in section 5. Some useful algebra can be found in the four appendices.

2. Preparation

The Dirac operator may be diagonalized with a complete set of orthonormal eigenfunctions: $i\not{D}\psi_n = i\gamma_\mu(\partial_\mu - igA_\mu)\psi_n = \lambda_n\psi_n$. This equation's Green's function is $G(x, y) = \sum_n \frac{\psi_n(x)\psi_n^\dagger(y)}{\lambda_n}$. If λ is an eigenvalue then $-\lambda$ is also an eigenvalue since $\{\gamma_5, i\not{D}\} = 0$. This also implies that the 2x2 matrix, G, is off-diagonal; so the trace of an odd number of them vanishes.

The eigenfunctions of $i\not{D}$ with zero eigenvalue are called zero modes. They span the on-shell part of the fermionic field and are dominant in the classical limit. In general the gauge

field may be written as $A_\mu = -\varepsilon_{\mu\nu}\partial_\nu(\phi + f) + \partial_\mu\rho$. The last term may be integrated over in the path integral with no consequence. The function f may be chosen to strictly carry any singularity of the electric field, called a vortex configuration, and ϕ is the quantum fluctuation around the vortex. They appear in the action as $\frac{1}{4}F^2 = \frac{1}{2}(\phi + f)\Box^2(\phi + f)$. From the index theorem

$$k = \frac{g}{2\pi}\int d^2x\ E = \frac{g}{2\pi}\int d^2x\ \varepsilon_{\mu\nu}\partial_\mu A_\nu = \frac{g}{2\pi}\int d^2x\ \Box f,$$

the background field, f, must satisfy $f(x \to \infty) = \frac{k}{g}\ \ln|x|$. Therefore the strength of the vortex is related to k, the number of right-handed minus the number of left-handed zero modes. The fact that this quantity is not zero is due to the anomaly.

It can be further shown [7] that the zero modes of a vortex are all of one chirality, hence $|k|$ is the total number of zero modes. This decomposition restricts integration of the gauge field on the plane in the path integral to integration over ϕ and a sum over all possible k. The three gamma matrices $(\gamma_0, \gamma_1, \gamma_5)$ can be chosen to be the Pauli matrices from which the identity $\varepsilon_{\mu\nu}\gamma_\nu\gamma_5 = i\gamma_\mu$ follows. Finally, solving $i\not{D}\chi = 0$ for a given flux on the plane gives

$$\begin{aligned} \chi_+^{(p)}(z, \overline{z}) &= N_{p,k}\, z^{p-1} e^{-gf(z,\overline{z})} \\ \chi_-^{(p)}(z, \overline{z}) &= N_{p,k}\, \overline{z}^{p-1} e^{gf(z,\overline{z})} \end{aligned}$$

with $z = x^0 + ix^1$, $p = 1, \ldots, |k|$, and $N_{p,k}$ is the normalization.

3. The Plane

The n-point scalar correlator (hereafter called the n-σ correlator) may be separated into sums of all possible combinations of the chiral operators:

$$\overline{\Psi}(x)\Big(\frac{1 \pm \gamma_5}{2}\Big)\Psi(x) \equiv \overline{\Psi}P_\pm\Psi(x) \tag{0.1}$$

and evaluated using the ideas of Bardacki and Crescimanno [5]. A general combination of these operators evaluated in a flux sector k will be notated by:

$$C^k_{e_1 \ldots e_n}(x_1, \ldots, x_n) \equiv \Big\langle \prod_{i=1}^n \overline{\Psi}P_{e_i}\Psi(x_i)\Big\rangle_k$$

where the e_i's can be either plus or minus. Since the Green's function is off-diagonal, terms only contribute for $n - k$ even. The transformation $\Psi(x) = e^{-g\gamma_5\phi(x)}\chi(x)$ separates the bosonic and fermionic integrals. Due to the chiral anomaly, this transformation has a non-trivial Jacobian $e^{m^2\int d^2x\phi\Box(\frac{1}{2}\phi + f)}$, which can be computed by building up infinitesimal transformations on Ψ (see Appendix A).

For the special case of $k = n$, only zero modes contribute to $C^n_{+\ldots+}$. In fact, the fermionic integral will vanish unless all the zero modes are taken into account here. A detailed calculation given in Appendix B gives

$$C^n_{+\ldots+}(x_1, \ldots, x_n) = M_{(n)} e^{2g^2 nK_{xx}} e^{4g^2 \sum\limits_{i>j} K_{x_i x_j}} \begin{vmatrix} 1 & z_1 & \ldots & z_1^{n-1} \\ 1 & z_2 & \ldots & z_2^{n-1} \\ \vdots & & & \vdots \\ 1 & z_n & \ldots & z_n^{n-1} \end{vmatrix}^2 =$$

$$= M_{(n)} e^{2g^2 n K_{xx}} e^{4g^2 \sum_{i>j} K_{x_i x_j}} \prod_{i>j=1}^{n} |z_i - z_j|^2.$$

The last equality can be proven by setting various z_i's and z_j's equal in the determinant and noting this is a zero since two lines are then equal. Here K_{xy} is the bosonic Green's function. It can be calculated by introduction of an infrared regulator, μ, which will be taken to zero at the end of the calculation.

$$K_{xy} = \frac{1}{\Box^2 - m^2 \Box} \delta^2(x-y) = \frac{1}{m^2} \Big(\frac{1}{\Box - m^2} - \frac{1}{\Box} \Big) \delta^2(x-y) =$$

$$= \lim_{\mu \to 0} \frac{1}{m^2} \Big(\frac{1}{\Box - m^2} - \frac{1}{\Box - \mu^2} \Big) \delta^2(x-y) = -\frac{1}{2g^2} [K_0(m|x-y|) + \ln \frac{\mu e^\gamma}{2} |x-y|]$$

$$K_{xx} = \frac{1}{2g^2} \ln \frac{m}{\mu}$$

Since μ does not depend on k, it is the same parameter for all Green's functions. Substituting in for all terms, the $|z_i - z_j|^2$ terms from the zero mode determinant cancel with the terms from the Green's functions giving

$$C^k_{+\ldots+}(x_1, \ldots, x_k) = M_{(k)} \frac{m^k}{\mu^{k^2}} \Big(\frac{2}{e^\gamma} \Big)^{k(k-1)} e^{-2 \sum_{i>j} K_0^{x_i x_j}} = C^{-k}_{-\ldots-}(x_1, \ldots, x_n). \qquad (0.2)$$

where $K_0^{xy} \equiv K_0(m|x-y|)$ and $M_{(k)} \equiv N_{1,k}^2 \ldots N_{k,k}^2$.

By taking $|x_i - x_j| \to \infty$ for all i and j, the cluster approximation reduces this correlator to a form only depending on the condensate [7], $\langle \overline{\Psi} \Psi \rangle = -m e^\gamma / 2\pi$

$$C^k_{+\ldots+}(x_1, \ldots, x_n) \;\to\; \Big(\frac{\langle \overline{\Psi} \Psi \rangle}{2} \Big)^k = M_{(k)} \frac{m^k}{\mu^{k^2}} \Big(\frac{2}{e^\gamma} \Big)^{k(k-1)},$$

and so a general form for $M_{(k)}$ may be found.

$$M_{(k)} = \Big(\frac{-1}{2\pi} \Big)^k \Big(\frac{\mu e^\gamma}{2} \Big)^{k^2} \qquad (0.3)$$

On the plane, the $p = k$ zero mode is non-normalizable. This takes the form of an infrared (large distance) divergence. However, this is exactly canceled by the infrared (small momentum) divergence of $(\Box - \mu^2)^{-1}$ as the regulator $\mu \to 0$ as seen in equation 0.3.

Calculation of non-minimal correlation functions requires knowledge of the fermionic Green's function [5] of $i\tilde{\not{D}} = i(\not{\partial} - g\gamma_5 \not{\partial} f)$ in arbitrary flux k. For convenience, we choose $k > 0$.

$$i\tilde{\not{D}} G(x,y) = \delta^2(x-y) - P(x,y)$$

$$\Rightarrow \begin{cases} 2i[\partial_z - g(\partial_z f)] G_{-+}(x,y) = \delta^2(x-y) - P(x,y) \\ 2i[\partial_{\overline{z}} + g(\partial_{\overline{z}} f)] G_{+-}(x,y) = \delta^2(x-y) \end{cases}$$

$$G_{-+}(x,x') = \langle 0 | \Psi_L(x) \overline{\Psi}_R(x') | 0 \rangle = \frac{e^{g[f(z,\overline{z}) - f(z',\overline{z}')]}}{2\pi i (\overline{z} - \overline{z}')}; \qquad G_{--} = 0$$

$$G_{+-}(x,x') = \langle 0 | \Psi_R(x) \overline{\Psi}_L(x') | 0 \rangle = \frac{e^{g[f(z',\overline{z}') - f(z,\overline{z})]}}{2\pi i (z - z')}; \qquad G_{++} = 0.$$

Here $P(x,y)$ is the projection operator onto the space of zero modes. The zero modes are all right handed and therefore $P(x,y)$ appears in only the G_{-+} equation. For $k<0$ the above equations would be exactly the same except that $P(x,y)$ would only appear in the G_{+-} equation. In fact, $P(x,y)$ may be ignored in the calculation of correlation functions since the product of zero modes and contractions are antisymmetrized [5] as expounded upon in Appendix C. In general, an analogue of Wick's theorem may be used to write out all contractions of Ψ_R's and Ψ_L's possible. The chiral neutral terms (like $\Psi_R\overline{\Psi}_L$) can be replaced by the corresponding Green's function above, and the purely chiral term with all the $\Psi_{\left\{ {R \atop L} \right\}}$'s (for $k=\pm|k|$) can be replaced by the zero modes.

Since $G_{+-}(x,y)=G^*_{-+}(y,x)$, the general non-minimal correlator may be written as (for $r-s=k>0$ and $r+s=n$):

$$C^k_{(+\ldots+)(-\ldots-)}(x_1,\ldots,x_r;x_{r+1},\ldots,x_{r+s}) = e^{2g^2nK_{xx}}e^{4g^2\sum\limits_{i>j}e_ie_jK_{x_ix_j}}e^{2g\sum\limits_{i=1}^{n}e_if(x_i)}\times$$

$$\times\begin{vmatrix} \chi^1_+(z_1,\overline{z}_1) & \ldots & \chi^k_+(z_1,\overline{z}_1) & G_{+-}(x_1,x_{r+1}) & \ldots & G_{+-}(x_1,x_{r+s}) \\ \vdots & & & & & \vdots \\ \chi^1_+(z_r,\overline{z}_r) & \ldots & \chi^k_+(z_r,\overline{z}_r) & G_{+-}(x_r,x_{r+1}) & \ldots & G_{+-}(x_r,x_{r+s}) \end{vmatrix}^2 \tag{0.4}$$

with $e_i=+$ for $1\le i\le r$ and $e_i=-$ for $r+1\le i\le r+s$. With this configuration,

$$\sum_{i>j}e_ie_j=\frac{(r-s)^2-(r+s)}{2}=\frac{k^2-n}{2}.$$

Writing out explicitly all the terms, equation (4) becomes

$$M_{(k)}\Big(\frac{m}{\mu}\Big)^n\Big(\frac{2}{\mu e^\gamma}\Big)^{k^2-n}\frac{1}{(2\pi)^{2s}}e^{-2\sum\limits_{i>j}e_ie_jK_0^{x_ix_j}}e^{2g\sum\limits_{i=1}^{n}e_if(x_i)}\prod_{i>j}|z_i-z_j|^{-2e_ie_j}\times$$

$$\times\begin{vmatrix} e^{-gf(x_1)} & z_1e^{-gf(x_1)} & \ldots & {z_1}^{k-1}e^{-gf(x_1)} & \frac{e^{g[f(x_{r+1})-f(x_1)]}}{z_1-z_{r+1}} & \ldots & \frac{e^{g[f(x_{r+s})-f(x_1)]}}{z_1-z_{r+s}} \\ \vdots & & & & & & \vdots \\ e^{-gf(x_r)} & z_re^{-gf(x_r)} & \ldots & {z_r}^{k-1}e^{-gf(x_r)} & \frac{e^{g[f(x_{r+1})-f(x_r)]}}{z_r-z_{r+1}} & \ldots & \frac{e^{g[f(x_{r+s})-f(x_r)]}}{z_r-z_{r+s}} \end{vmatrix}^2$$

It is now easy to see how the f dependence from the determinant cancels the extra term from the bosonic integration. Also, as in the minimal correlator above, the $|z_i-z_j|^2$ terms cancel as well. Since the only allowed k are such that $(-1)^k=(-1)^n$, the remaining terms may be written as:

$$C^k_{(+\ldots+)(-\ldots-)}(x_1,\ldots,x_n)=\left(-\frac{me^\gamma}{4\pi}\right)^n e^{-2\sum\limits_{i>j}e_ie_jK_0^{x_ix_j}}.$$

This can be inserted into the correlators to give

$$\begin{aligned}\left\langle\overline{\Psi}\Psi(x)\overline{\Psi}\Psi(y)\right\rangle &= \langle\overline{\Psi}\Psi\rangle^2\cosh(2K_0^{xy})\\ \left\langle\overline{\Psi}\Psi(x)\overline{\Psi}\Psi(y)\overline{\Psi}\Psi(z)\right\rangle &= \langle\overline{\Psi}\Psi\rangle^3\frac{1}{4}(e^{2(K_0^{xz}+K_0^{yz}-K_0^{xy})}+e^{2(K_0^{yz}+K_0^{xy}-K_0^{xz})}+\end{aligned}$$

$$
\begin{aligned}
& +e^{2(K_0^{xy}+K_0^{xz}-K_0^{yz})}+e^{-2(K_0^{xy}+K_0^{xz}+K_0^{yz})}) \\
= \; & \langle\overline{\Psi}\Psi\rangle^3[\cosh(2K_0^{xy})\cosh(2K_0^{xz})\cosh(2K_0^{yz})+ \\
& -\sinh(2K_0^{xy})\sinh(2K_0^{xz})\sinh(2K_0^{yz})]
\end{aligned}
$$

$$
\Big\langle\prod_{i=1}^{n}\overline{\Psi}\Psi(x_i)\Big\rangle = \Big(\frac{\langle\overline{\Psi}\Psi\rangle}{2}\Big)^n \sum_{\{e_i\}\in\{\pm\}} e^{-2\sum\limits_{i>j}e_ie_jK_0^{x_ix_j}}. \tag{0.5}
$$

The last equation, (0.5) is the general n-σ correlator. The sum for the sets of e_i's runs over all possible combinations of pluses and minuses. It is interesting to note that this expression can be viewed as the partition function of a two dimensional system of n fermions at fixed points with charges ± 1 and potential K_0^{xy} which at small distances reduces to the potential studied by Kosterlitz and Thouless [8]. This observation was originally due to Samuel [10].

4. The Torus

We would like to study the Schwinger model at finite temperature. Placing the model on a torus [7] with sides $[0,L]\times[0,\beta]$ gives the appropriate geometry for such studies. Finally, we shall impose the thermodynamic limit by taking L to infinity and obtain the finite temperature result. Taking β to infinity should reproduce the results obtained on the plane for zero temperature.

The zero modes are more complicated here than on the plane (Jacobi theta functions) yet are normalizable in the finite volume.

$$
\varphi_p(x) = \Big(\frac{2|k|}{\beta^2 V}\Big)^{\frac{1}{4}} U(x)\, e^{\mp g[\phi(x)+f(x)]}\,\Theta\Big[\begin{matrix}(p-\frac{1}{2}-h_0)/k \\ h_1\end{matrix}\Big](kz, i|k|\tau);
$$

$$
\text{with}\quad f(x)=\frac{\pi k}{V}(x^1)^2, \quad U(x)=e^{2\pi i[h_0x^0/\beta+h_1x^1/L]},
$$

$z=(x^0+ix^1)/\beta$, and $\tau=L/\beta$ similar to the notation of Sachs and Wipf [7]. The singularity that $\Box f$ had on the plane is spread out over the torus so as not to ruin the periodicity. The A integration may be replaced by a sum over k and integration over ϕ as in Appendix B for the plane. In addition, h_0 and h_1 are the 2π periodic degrees of freedom gained by changing the topology of the space and they must also be integrated over:

$$
\int DA_\mu\ldots = \sum_k \langle\ldots\rangle_A = \sum_k \int d^2h D\phi \ldots e^{-\frac{1}{4}\int F^2} =
$$
$$
= \sum_k \int d^2h D\phi\ldots e^{-\frac{1}{2}\int\phi\Box^2\phi} e^{-2\pi k^2/m^2V}.
$$

The h's are defined modulo 1 corresponding to a full revolution around the two periodicities of the torus.

The 3-σ correlator may be broken up into four distinct terms considering the equivalent contributions from the positive and negative k sectors.

$$
\Big\langle\overline{\Psi}\Psi(x)\overline{\Psi}\Psi(y)\overline{\Psi}\Psi(z)\Big\rangle = 2[C^3_{+++}(x,y,z)+C^1_{++-}(x,y,z)+C^1_{++-}(y,z,x)+C^1_{++-}(z,x,y)].
$$

The general form of these terms may be written conveniently as (for $r-s=k>0$):

$$C^k_{(+\ldots+)(-\ldots-)}(x_1,\ldots,x_r;x_{r+1},\ldots,x_{r+s}) =$$

$$= Z^{-1}\Big\langle \det{}'(i\not{D}_k) \begin{vmatrix} \varphi_1(x_1) & \ldots & \varphi_k(x_1) & S_{+-}(x_1,x_{r+1}) & \ldots & S_{+-}(x_1,x_{r+s}) \\ \vdots & & & & & \vdots \\ \varphi_1(x_r) & \ldots & \varphi_k(x_r) & S_{+-}(x_r,x_{r+1}) & \ldots & S_{+-}(x_r,x_{r+s}) \end{vmatrix}^2 \Big\rangle_A.$$

The finite temperature Green's function in the presence of flux k satisfies the equation:

$$i\not{D}S(x,y) = \delta^2(x,y) - P(x,y).$$

Appendix C shows that the solution can be written as just the $k=0$ Green's function [9] with an h-independent term multiplying it to ensure correct boundary conditions.

$$S_{+-}(x,y) = S^*_{-+}(y,x) =$$

$$= \frac{1}{2\pi i\beta} e^{g(\phi(y)-\phi(x)+f(y)-f(x))} U(x)U^\dagger(y) e^{-2\pi i h_0(z_1-z_2)} \vartheta_1'(0) T_k(z_1,z_2)$$

$$\text{where}\quad T_k(z_1,z_2) = \frac{\vartheta_3(kz_1|ik\tau)\vartheta_4(z_1-z_2+H)}{\vartheta_3(kz_2|ik\tau)\vartheta_4(H)\vartheta_1(z_1-z_2)} \quad\text{and}\quad H = h_1 - i\tau h_0.$$

These theta functions are related to those from Sachs and Wipf by:

$$\Theta\begin{bmatrix} a \\ b \end{bmatrix}(z,i\tau) = e^{2\pi i a(z+b+\frac{i}{2}\tau a)} \vartheta_3(\pi(z+b+i\tau a)|i\tau)$$

and the π in the argument and the dependence on τ has been suppressed for ease in reading.

Continuing the calculation of the 3-σ correlator, it may be simplified after the ϕ integration to

$$C^1_{++-}(x,y,z) = \frac{\sqrt{2\tau}|\eta(i\tau)|^8}{\beta^3} e^{6g^2K_{xx}+4g^2(K_{xy}-K_{xz}-K_{yz})} e^{-2\pi/m^2V} \times$$

$$\times \int d^2h e^{-2\pi\tau {h_0}^2 + 4\pi h_0(x^1+y^1-z^1)/\beta} \begin{vmatrix} \vartheta_2(z_1+H) & T_1(z_1,z_3) \\ \vartheta_2(z_2+H) & T_1(z_2,z_3) \end{vmatrix}^2 .$$

Here K_{xy} is the bosonic Green's function as before on the plane. Its explicit form on the torus is (see Appendix D)

$$4g^2K_{xy} = 4\pi\Delta_m(x-y) + \ln\left|\frac{\eta(i\tau)}{\vartheta_1(z_1-z_2)}\right|^2 + \frac{2\pi}{V}(x^1-y^1)^2 + \frac{4\pi}{m^2V}.$$

After also identifying the finite temperature condensate [7]:

$$\langle\overline{\Psi}\Psi\rangle_\beta = -\frac{2|\eta(i\tau)|^2}{\beta} e^{-2\pi/m^2V} e^{2g^2K_{xx}},$$

the final form for finite volume is

$$C^1_{++-}(x,y,z) = \left(\frac{\langle\overline{\Psi}\Psi\rangle_\beta}{2}\right)^3 e^{4\pi[\Delta_m(x-y)-\Delta_m(x-z)-\Delta_m(y-z)]} F_{++-}(x,y,z) \tag{0.6}$$

with

$$F_{++-}(x,y,z) = \frac{\sqrt{2\tau}}{|\vartheta_1(z_1-z_2)\vartheta_3(z_3)|^2} \int \frac{d^2h}{|\vartheta_4(H)|^2} e^{-2\pi(x^1+y^1-z^1-Lh_0)^2/V} \times$$
$$\times \left| \begin{matrix} \vartheta_2(z_1+H) & \vartheta_3(z_1)\vartheta_4(z_1-z_3+H)\vartheta_1(z_2-z_3) \\ \vartheta_2(z_2+H) & \vartheta_3(z_2)\vartheta_4(z_2-z_3+H)\vartheta_1(z_1-z_3) \end{matrix} \right|^2 .$$

The full massive bosonic Green's function, $2\pi\Delta_m(x-y)$ may be rewritten in the limit of $L\to\infty$ giving :

$$-2\pi\Delta_m(x) = \sum_{k=-\infty}^{\infty} K_0(m\sqrt{(x^0-\beta k)^2+(x^1)^2})$$

which shows the zero temperature contribution is just $K_0(m|x|)$ as obtained on the plane. There is also an extra topological factor, F_{++-}, which is not able to be exactly integrated, but as $L\to\infty$:

$$\vartheta_1(z) \to 2e^{-\frac{\pi\tau}{4}}\sin\pi z \qquad \vartheta_2(z+H) \to e^{-\pi\tau(\frac{1}{4}-h_0)}e^{\pi i(z+h_1)}$$
$$\vartheta_3(z) \to 1 \qquad \vartheta_4(z+H) \to 1 - e^{\pi\tau(2h_0-1)}e^{2\pi i(z+h_1)}.$$

and $F_{++-}(x,y,z) \to 1$. Taking $\beta\to\infty$ gives the same result as on the plane, thereby confirming our results.

Considering ways in which the general result could depend on β such that the zero temperature limit is equation (0.5) and the 3-σ correlator is equation (0.6) compel us to one conclusion. This is that the finite temperature result can be obtained from the zero temperature result by simply replacing the zero temperature condensate and Green's function with their finite temperature counterparts. If this is indeed the case, the n-σ finite temperature correlator can then be written as

$$\left\langle\prod_{i=1}^n \overline{\Psi}\Psi(x_i)\right\rangle = \left(\frac{\langle\overline{\Psi}\Psi\rangle_\beta}{2}\right)^n \sum_{\{e_i\}\in\{\pm\}} e^{4\pi\sum_{i>j} e_i e_j \Delta_m(x_i-x_j)}.$$

In the bosonized form of the Schwinger model [11] the structure of this result is expected [12]. It is reassuring that the exact finite temperature calculation done here confirms this, giving us some confidence in the generalization procedure at finite temperature and infinite volume. The generalization to finite temperature and finite volume is less straightforward.

5. Conclusion

Two important ways to calculate the n-σ correlation functions in the Schwinger model were looked at specifically to find a general form for the n-point scalar correlators. The plane is easier to work on due to the absence of the cumbersome theta functions, but the validity of results from the plane have been questioned in the past. Therefore the result

was also derived on the torus and shown to be equivalent to the plane in the large volume limit.

In fact, both methods are very similar. The large distance divergence in the zero modes are canceled by the small momentum divergence in the bosonic Green's function. The final result for the finite temperature n-σ correlator is identical to the zero temperature result if the zero temperature condensate and Green's function are replaced by their finite temperature counterparts. There is no other spurious temperature dependence as could be expected due to the differing topologies in which the two problems were solved.

The results presented here are directly amenable to the spectral analysis discussed by Fayyazuddin *et al.* [9]. They show that in the Schwinger model, the four-, six-, ..., n-point scalar correlation functions have a screening length that is determined by the meson mass $m = e/\sqrt{\pi}$ and independent of the temperature. This is not the case of the two-point gauge invariant fermion correlator, where the screening mass was found to asymptote $\pi T/2$ due to the emergence of two-meron configurations at high temperature [15]. While the present calculations are schematic and far from the real world, they still provide nontrivial insights into nonperturbative physics at finite temperature. They are certainly of some interest for comparison with finite temperature lattice simulations of the model [13].

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Appendix A

The non-invariance of the fermionic measure under the chiral transformation $\Psi(x) = e^{-g\gamma_5\phi(x)}\chi(x)$ leads to a non-trivial Jacobian. First noticed by Fujikawa in four dimensions [14], we give a derivation in the two dimensional case because there are certain subtleties and we have not seen an explicit calculation in the literature.

Expanding $\Psi(x)$ and $\chi(x)$ in orthonormal eigenmodes of the Dirac operator, $\varphi_n(x)$, the above transformation for an infinitesimal change, $\delta\phi(x)$, in the quantum fluctuation field gives

$$D\overline{\Psi}D\Psi = \exp\Big[-\int d^2x\,\delta\phi(x)\,\sum_n \varphi_n^\dagger(x)\gamma_5\varphi_n(x)\Big]D\overline{\chi}D\chi =$$

$$= e^{m^2\int d^2x\,\delta\phi\,\varepsilon_{\mu\nu}\partial_\mu A_\nu}D\overline{\chi}D\chi$$

for the measure. However, $A_\nu = -\varepsilon_{\nu\rho}\partial_\rho(\delta\phi + f)$ and so

$$\delta\phi\varepsilon_{\mu\nu}\partial_\mu A_\nu = \delta\phi\Box(\delta\phi + f).$$

Integrating the infinitesimal transformations, the additional term in the Lagrangian for finite ϕ is $m^2\int d^2x\phi\Box(\frac{1}{2}\phi + f)$.

Appendix B

The n-σ correlator may be broken into s_+ and s_- parts and simplified by the chiral transformation discussed in Appendix A to separate the fermionic and bosonic integrations.

$$C^k_{e_1\ldots e_n}(x_1,\ldots,x_n) = Z^{-1}\int D\overline{\chi}D\chi DA\prod_{i=1}^n \overline{\chi}P_{e_i}\chi(x_i)\,e^{-2g\sum e_i\phi(x_i)}\,e^{-S_{eff}}$$

$$= Z^{-1}\sum_k\int D\overline{\chi}D\chi D\phi\prod_{i=1}^n \overline{\chi}P_{e_i}\chi(x_i)\,e^{-\int d^2x(-\overline{\chi}i\tilde{\not{D}}\chi+J\phi+\frac{1}{2}f\Box^2 f)}\,e^{-\frac{1}{2}\int d^2x\phi(\Box^2-m^2\Box)\phi}$$

$$\text{where}\quad S_{eff} = \int d^2x\Big[-\overline{\chi}i\tilde{\not{D}}\chi + \frac{1}{2}(\phi+f)(\Box^2-m^2\Box)(\phi+f)+\frac{m^2}{2}f\Box f\Big]$$

$$J = (\Box^2-m^2\Box)f + 2g\sum_{i=1}^n e_i\delta^2(x-x_i)\qquad\text{and}\qquad \tilde{\not{D}} = \not{\partial} - g\gamma_5\not{\partial}f.$$

Now completing the square in the bosonic integral (with $B=\Box^2-m^2\Box$):

$$\frac{1}{2}\phi B\phi + J\phi = \frac{1}{2}(\phi+\frac{J}{B})B(\phi+\frac{J}{B}) - \frac{1}{2}JB^{-1}J.$$

Therefore the bosonic Gaussian cancels the same factor in the denominator. Further calculation yields:

$$\frac{1}{2}\int d^2xJB^{-1}J = \frac{1}{2}\int d^2xfBf + 2g\sum_{i=1}^n e_if(x_i) + 2g^2\sum_{i,j}^n e_ie_jK_{x_ix_j}$$

$$= \frac{1}{2}\int d^2xf(\Box^2-m^2\Box)f + 2g\sum_{i=1}^n e_if(x_i) + 2g^2nK_{xx} + 4g^2\sum_{i>j}^n e_ie_jK_{x_ix_j}.$$

Antisymmetrizing the fermionic functions:

$$\sum_\sigma(-1)^\sigma\prod_{p=1}^k\chi_R^{(p)}(x_{\sigma(p)}) = M^{1/2}_{(k)}e^{-g\sum f(x_i)}\begin{vmatrix} 1 & z_1 & \ldots & z_1^{k-1}\\ 1 & z_2 & \ldots & z_2^{k-1}\\ \vdots & & & \vdots\\ 1 & z_k & \ldots & z_k^{k-1}\end{vmatrix} =$$

$$= M^{1/2}_{(k)}e^{-g\sum f(x_i)}\prod_{i>j=1}^k(z_i-z_j).$$

The non-zero mode fermionic integration leads to a factor:

$$\frac{\det'(i\tilde{\not{D}}_k)}{\det(i\not{\partial})} = \exp\Big[\frac{1}{2}\mathrm{Tr}\ln\Big(\frac{-\tilde{\not{D}}^2}{-\not{\partial}^2}\Big)\Big] = e^{\frac{m^2}{2}\int d^2xf\Box f}$$

since

$$\mathrm{Tr}\ln\Big(\frac{-\tilde{\not{D}}^2}{-\not{\partial}^2}\Big) = -\mathrm{Tr}\int_0^1 d\alpha\frac{d}{d\alpha}\int_0^\infty\frac{ds}{s}e^{s\tilde{\not{D}}^2_\alpha} = m^2\int d^2xf\Box f$$

where $\tilde{D\!\!\!/}_\alpha = e^{\alpha g \gamma_5 f}\,\partial\!\!\!/\,e^{\alpha g\gamma_5 f}$.

Gathering all terms together, the zero mode saturated correlator becomes

$$\begin{aligned}\Big\langle \prod_{i=1}^{k} s_+(x_i)\Big\rangle_k &= M_{(k)} \prod_{i>j} |x_i - x_j|^2\, e^{2g^2 k K_{xx}}\, e^{4g^2 \sum\limits_{i>j} K_{x_i x_j}} \\ &= M_{(k)} \frac{m^k}{\mu^{k^2}} \Big(\frac{2}{e^\gamma}\Big)^{k(k-1)} e^{-2\sum\limits_{i>j} K_0(m|x_i - x_j|)}.\end{aligned}$$

where $|z_i - z_j| = |x_i - x_j|$ was used.

Appendix C

We derive the fermionic Green's function in the presence of a non-trivial flux k on the torus. This is required for the non-minimal correlation functions at finite temperature. Although other authors have tackled this problem [16], a more explicit form is needed for practical calculations. The Green's function is defined by:

$$iD\!\!\!/ S(x,y) = \delta^2(x-y) - P(x,y)$$

where $P(x,y)$ is the projection operator onto the space of zero modes. For convenience, k will be taken positive.

Using the trivialization of the $U(1)$ bundle on the torus as in Sachs and Wipf [7], the gauge potential is explicitly:

$$\begin{aligned} A_0 &= -\frac{\Phi}{V} x^1 + \frac{2\pi}{\beta} h_0 - \partial_1 \phi \\ A_1 &= \frac{2\pi}{L} h_1 + \partial_0 \phi. \end{aligned}$$

Noticing that:

$$iD\!\!\!/ = e^{g\gamma_5[\phi(x)+f(x)]} U(x) i\partial\!\!\!/ U^\dagger(x) e^{g\gamma_5[\phi(x)+f(x)]}, \tag{0.7}$$

we can write

$$S(x,y) = e^{-g\gamma_5[\phi(x)+f(x)]} U(x) g(x,y) U^\dagger(y) e^{-g\gamma_5[\phi(y)+f(y)]}. \tag{0.8}$$

This reduces our problem to solving the equation:

$$i\partial\!\!\!/ g(x,y) = \delta^2(x-y) - \tilde{P}(x,y)$$

$$\text{where} \quad \tilde{P}(x,y) = e^{g\gamma_5[\phi(x)+f(x)]} U^\dagger(x) P(x,y) U(y) e^{g\gamma_5[\phi(y)+f(y)]}.$$

Writing the components of this matrix equation out and letting $z = (x^0 + ix^1)/\beta$ and $w = (y^0 + iy^1)/\beta$,

$$\begin{aligned} -\beta \partial_z g_{-+}(z-w) &= \delta^2(z-w) - \tilde{P}(z,w) \\ -\beta \partial_{\bar{z}} g_{+-}(z-w) &= \delta^2(z-w). \end{aligned}$$

Following the ideas of [5], we realize that g_{+-} and g_{-+} both appear as contractions of left- and right-handed fermionic fields in calculations of the non-minimal correlation functions. Writing

$$g_{-+}(z,w) = \tilde{g}_{-+}(z,w) - \sum_{n=1}^{k} \chi_n(z)\eta_n^*(w)$$

$$\text{with} \qquad \sum_{n=1}^{k} \eta_n(z)\eta_n^*(w) = \tilde{P}(z,w),$$

$$-\beta\partial_z\chi_n(z) = \eta_n(z), \quad \text{and} \quad -\beta\partial_z\tilde{g}_{-+}(z,w) = \delta^2(z-w).$$

The summation above is a summation of the zero modes. In the calculation of non-minimal correlation functions, the product of zero modes and contractions needs to be antisymmetrized. As shown in section **4**, the Green's function appears along with all of the zero modes in a determinant. For this reason, the extra summation above will not contribute. Therefore only $\tilde{g}_{+-}$ and $\tilde{g}_{-+}$ need to be calculated.

In the presence of a non-zero flux, fields at (x^0, x^1) and $(x^0, x^1 + L)$ are related by the gauge transformation:

$$A_\mu(x^0, x^1 + L) - A_\mu(x^0, x^1) = \partial_\mu \alpha, \qquad \Psi(x^0, x^1 + L) = e^{i\alpha}\Psi(x^0, x^1)$$
$$\text{where} \qquad \alpha = -\frac{2\pi k}{\beta} x^0. \tag{0.9}$$

From equations 0.8 and 0.9, the boundary conditions on $\tilde{g}$ can be found to be:

$$\tilde{g}_{+-}(z+1, w) = -e^{-2\pi i h_0}\tilde{g}_{+-}(z,w) \qquad \tilde{g}_{+-}(z, w+1) = -e^{2\pi i h_0}\tilde{g}_{+-}(z,w)$$
$$\tilde{g}_{+-}(z + i\tau, w) = e^{\pi k\tau} e^{-2\pi i(kz+h_1)}\tilde{g}_{+-}(z,w) \tag{0.10}$$
$$\tilde{g}_{+-}(z, w + i\tau) = e^{-\pi k\tau} e^{2\pi i(kw+h_1)}\tilde{g}_{+-}(z,w).$$

The general form of $\tilde{g}_{+-}$ is

$$\tilde{g}_{+-}(z,w) = \frac{1}{2\pi i\beta}\frac{\vartheta_1'(0)}{\vartheta_1(z-w)} f(z,w)$$

and $f(z,w)$ must be chosen to complete the boundary conditions in equation 0.10, have no poles for $z = w$, and be 1 at $z = w$. A solution that satisfies these conditions is

$$f(z,w) = \frac{\vartheta_4(z-w+H)}{\vartheta_4(H)}\frac{\vartheta_3(kz|ik\tau)}{\vartheta_3(kw|ik\tau)} e^{-2\pi i h_0(z-w)}.$$

A similar calculation can be done for $\tilde{g}_{-+}$ giving finally:

$$\tilde{g}_{+-}(z,w) = \frac{1}{2\pi i\beta}\frac{\vartheta_1'(0)\vartheta_4(z-w+H)\vartheta_3(kz|ik\tau)}{\vartheta_1(z-w)\vartheta_4(H)\vartheta_3(kw|ik\tau)} e^{-2\pi i h_0(z-w)}$$

$$\tilde{g}_{-+}(z,w) = \frac{1}{2\pi i\beta}\frac{\vartheta_1'(0)\vartheta_4(\overline{z}-\overline{w}-\overline{H})\vartheta_3(k\overline{w}|ik\tau)}{\vartheta_1(\overline{z}-\overline{w})\vartheta_4(\overline{H})\vartheta_3(k\overline{z}|ik\tau)} e^{-2\pi i h_0(\overline{z}-\overline{w})}.$$

Inserting this in equation 0.8, we obtain the fermionic Green's function.

The reader might be alarmed that we seem to have managed to invert the Dirac operator which is singular in the presence of a non-zero flux. If the solution is not unique, it will differ from the one above by a linear superposition of zero modes. This additional term does not contribute in the calculation of the correlation function by exactly the same arguments presented earlier for going from g_{-+} to $\tilde{g}_{-+}$.

Appendix D

The bosonic Green's function, K_{xy}, may be broken up into a massive and massless part:

$$K_{xy} = \langle x| \frac{1}{\Box(\Box - m^2)} |y\rangle = \frac{1}{m^2} \langle x| \frac{1}{\Box - m^2} - \frac{1}{\Box} |y\rangle =$$

$$= \frac{1}{m^2}(\Delta_m(x-y) + \frac{1}{m^2 V} - \Delta(x-y))$$

as done in [9].

The massless part may be calculated by using the eigenmodes that span the allowed field ϕ on the torus.

$$\Delta(x-y) = \langle x| \frac{1}{\Box} |y\rangle = -\sum_{n\neq 0} \frac{\phi_n(x)\phi_n^\dagger(y)}{\mu_n}$$

$$\text{with} \quad \phi_n(x) = \frac{1}{\sqrt{V}} \phi_n e^{2\pi i (n_0 x^0/\beta + n_1 x^1/L)} \quad \text{and} \quad \phi_n \phi_m^\dagger = \delta_{nm}$$

Since $\int \phi = 0$ is a necessary condition for a one to one transformation of the fields, the value $n = (n_0, n_1) = 0$ is omitted from the summation. The eigenvalues due to the periodicity of the torus are

$$\mu_n = \Big(\frac{2\pi n_0}{\beta}\Big)^2 + \Big(\frac{2\pi n_1}{L}\Big)^2$$

as found in Sachs and Wipf [7].

The sum may be broken up into two terms: one with $n_0 \neq 0$ and the other with $n_0 = 0$ and $n_1 \neq 0$. Evaluating the first n_1 sum and relabeling the remaining summation index as n,

$$\Delta(x) = -\frac{1}{2\pi} \sum_{n>0} \Big\{ \frac{\cos(2\pi n \frac{x^0}{\beta})}{n} \Big[\coth(\pi\tau n)\cosh(2\pi n \frac{x^1}{\beta}) - \sinh(2\pi n \frac{x^1}{\beta})\Big] + \frac{\tau}{\pi} \frac{\cos(2\pi n \frac{x^1}{L})}{n^2} \Big\}.$$

Letting $z = (x^0 + ix^1)/\beta$ and using the identities:

$$\cos(kx)\cosh(ky) = \mathrm{Re}[\cos k(x+iy)] \quad \text{and} \quad \cos(kx)\sinh(ky) = \mathrm{Re}[-i\sin k(x+iy)],$$

the first summation in the braces may be written as:

$$\mathrm{Re} \sum_{n>0} \Big[\frac{\cos 2\pi n z}{n} (\coth(\pi\tau n) - 1) + \frac{e^{2\pi i n z}}{n} \Big] =$$

$$= \mathrm{Re} \sum_{n,r>0} \Big[\frac{e^{2\pi i n z} + e^{-2\pi i n z}}{n} e^{-2\pi\tau n r} \Big] - \ln(1 - e^{2\pi i z}) =$$

$$= -\mathrm{Re} \ln \Big\{ 2ie^{\pi i z} \sin(\pi z) \prod_{r>0} (1 - 2e^{-2\pi\tau r} \cos(2\pi z) + e^{-4\pi\tau r}) \Big\}$$

since $\coth(\pi\tau n) - 1 = \sum_{r>0} e^{-2\pi\tau n r}$ and $\ln(1-x) = -\sum_{n>0} \frac{x^n}{n}$.

The second summation in the braces gives

$$\frac{\pi\tau}{6} - \pi x^1 + \frac{\pi (x^1)^2}{\tau}.$$

The first two terms can be combined with the first summation to form the product representation of ϑ_1 and $\eta(i\tau)$ to finally give

$$-4\pi\Delta(x-y) = \ln \Big| \frac{\eta(i\tau)}{\vartheta_1(\pi(z_1 - z_2)|i\tau)} \Big|^2 + \frac{2\pi}{V}(x^1 - y^1)^2.$$

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